19980707 127

Eusipco 98, Greece

RESULTS ON CYCLIC SIGNAL PROCESSING SYSTEMS

P. P. Vaidyanathan

Dept. Electrical Engr., Caltech, Pasadena, CA 91125

Abstract

We present a state space description for cyclic LTI systems which find applications in cyclic filter banks and wavelets. We also revisit the notions of reachability and observability in the cyclic context, and show a number of important differences from traditional noncyclic case. A number of related problems such as the paraunitary interpolation problem and the cyclic paraunitary factorizability problem can be understood in a unified way by using the realization matrix defined by the state space description.

1. INTRODUCTION

Cyclic digital filters and filter banks have recently been introduced in the signal processing literature. In particular, the fundamentals of cyclic multirate systems are introduced in [**1**,**9**], and the concepts applied to cyclic filter banks. Specific detailed problems pertaining to wavelet construction [**2**,**3**] have also been recently addressed. The applications of cyclic filter banks in image coding has been pointed out in [**6**,**7**]. In this paper we concentrate on state space descriptions of cyclic LTI systems and point out several departures from conventional state space theory. We also revisit the cyclic paraunitary interpolation problem [**10**] and the cyclic paraunitary factorization problem from a state space viewpoint and show a common link between these.

The input-output description for a cyclic(L) LTI system is a circular convolution

$$\mathbf{y}(n) = \sum_{m=0}^{L-1} \mathbf{h}(m)\mathbf{x}(n-m)$$

where the time arguments are interpreted modulo L. The frequency response matrix $\mathbf{H}(k)$ is given by the L-point DFT of the impulse response matrix, that is $\mathbf{H}(k) = \sum_{n=0}^{L-1} \mathbf{h}(n)W_L^{kn}$ where $W_L = e^{-2\pi j/L}$. The quantity W_L^k is interpreted as a unit of "cyclic-delay", and is used in structures representing cyclic systems. Any cyclic LTI system can be drawn in nonrecursive form using $L-1$ cyclic delays as in Fig. 1, but sometimes recursive structures are more economic [**9**]. The **recursive cyclic structure** in Fig. 2 has the frequency response $H(k) = (a_0 + a_1 W_L^k)/(1 - bW_L^k)$. Implementing such a recursive structure brings up the question of initial conditions, which are tricky because of the cyclic nature of time. We will address this in the more general setting of state space descriptions.

An $M \times M$ cyclic transfer matrix $\mathbf{E}(k)$ is said to be **paraunitary** if it is unitary for $0 \le k \le L-1$. This finds application in **cyclic orthonormal** filter banks [**9**]. Unlike their noncyclic counterparts it has been shown in [**10**] that cyclic paraunitary matrices are not always factorizable. We return to this in Sec. 3.

[1]Work supported in parts by the Office of Naval Research Grant N00014-93-1-0231, and Tektronix, Inc.

2. CYCLIC STATE-SPACE DESCRIPTIONS

Consider a cyclic LTI structure with N cyclic delay elements (e.g., $N = L-1$ and $N = 1$ in Figs. 1 and 2, respectively). We can identify a set of N state variables $v_i(n)$ (outputs of the unit delay elements W_L^k) and obtain equations of the form

$$\mathbf{v}(n+1) = \mathbf{A}\mathbf{v}(n) + \mathbf{B}\mathbf{x}(n) \qquad (1(a))$$

$$\mathbf{y}(n) = \mathbf{C}\mathbf{v}(n) + \mathbf{D}\mathbf{x}(n) \qquad (1(b))$$

where $\mathbf{v}(n) = [v_1(n) \quad v_2(n) \quad \ldots \quad v_N(n)]^T$ is the state vector. Since this system can have multiple inputs and outputs, we have used bold letters $\mathbf{x}(n)$ and $\mathbf{y}(n)$ above. Repeated use of (1(a)) yields $\mathbf{v}(L) = \mathbf{A}^L\mathbf{v}(0)$ + a linear combination of samples of $\mathbf{x}(n)$. Since all the time-indices are interpreted modulo-L, we have $\mathbf{v}(L) = \mathbf{v}(0)$, and

$$(\mathbf{I} - \mathbf{A}^L)\mathbf{v}(0) = \text{linear combination of samples of } \mathbf{x}(n).$$

Thus we can identify the initial state $\mathbf{v}(0)$, provided $\mathbf{I} - \mathbf{A}^L$ is nonsingular, i.e., no eigenvalue of $\mathbf{A}$ has the form W_L^m for any integer m. In other words, the eigenvalues of $\mathbf{A}$ should not be at the unit-circle points indicated in Fig. 3. This nonsingularity condition can be understood in another way. If we evaluate the frequency response $\mathbf{H}(k)$ explicitly, we have the form

$$\mathbf{H}(k) = \mathbf{D} + \mathbf{C}\Big(W_L^{-k}\mathbf{I} - \mathbf{A}\Big)^{-1}\mathbf{B} = \frac{\mathbf{P}(k)}{\det\,(W_L^{-k}\mathbf{I} - \mathbf{A})}$$

The eigenvalue condition on $\mathbf{A}$ implies that the denominator $\det\,(W_L^{-k}\mathbf{I} - \mathbf{A})$ is nonzero for all integers k. As long as this is satisfied, $\mathbf{H}(k)$ is defined for all k, and we can uniquely identify an "initial state" $\mathbf{v}(0)$ for any input sequence $\{\mathbf{x}(n)\}$.

Even though the expression for $\mathbf{H}(k)$ resembles the noncyclic case $\mathbf{H}_{non}(z) = \mathbf{D} + \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$, the impulse response matrix $\mathbf{h}(n)$ takes a slightly different form:

$$\mathbf{h}(n) = \begin{cases} \mathbf{D} + \mathbf{C}\mathbf{A}^{L-1}\Big(\mathbf{I} - \mathbf{A}^L\Big)^{-1}\mathbf{B} & n = 0 \\ \mathbf{C}\mathbf{A}^{n-1}\Big(\mathbf{I} - \mathbf{A}^L\Big)^{-1}\mathbf{B} & 1 \le n \le L-1 \end{cases} \qquad (2)$$

Notice, for example, that $\mathbf{h}(0) \neq \mathbf{D}$, which is a departure from the noncyclic case. These differences arise because the initial condition $\mathbf{v}(0)$ is predetermined as explained earlier,

DTIC QUALITY INSPECTED 1

DISTRIBUTION STATEMENT A
Approved for public release;
Distribution Unlimited

and cannot be set to zero (as we would in the noncyclic case). Though the computation of $\mathbf{v}(0)$ in general requires an initial overhead, such a computation followed by the recursive computation of $\mathbf{y}(n)$ as in Eq. (1) is often more economic than direct or FFT-based circular convolution of $\mathbf{x}(n)$ and $\mathbf{h}(n)$.

Similarity transformations. If we define a new state space description by using the familiar similarity transform $\mathbf{A}_1 = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}, \mathbf{B}_1 = \mathbf{T}^{-1}\mathbf{B}, \mathbf{C}_1 = \mathbf{C}\mathbf{T}$, the new system $(\mathbf{A}_1, \mathbf{B}_1, \mathbf{C}_1, \mathbf{D})$ has the same $\mathbf{h}(n)$. Reason: we can verify by substitution that $\mathbf{C}\mathbf{A}^{n-1}\left(\mathbf{I}-\mathbf{A}^L\right)^{-1}\mathbf{B}$ is unchanged by the similarity transform for any $n \geq 1$. Thus we can find equivalent cyclic state space realizations by using similarity transforms. Note that $\mathbf{D}$ is unchanged by the transform.

2.1. Reachability

The ideas of reachability and observability [5], [8] can be extended to cyclic LTI systems but there are some differences from the traditional noncyclic case. For example we will see that reachability and observability together do not imply minimality. The cyclic LTI system is said to be reachable if we can arrive at any chosen final value $\mathbf{v}_f$ for the state vector $\mathbf{v}(n)$ at any chosen time n by proper choice of the input sequence $\mathbf{x}(.)$. To quantify this consider the state recursion $\mathbf{v}(n+1) = \mathbf{A}\mathbf{v}(n) + \mathbf{B}\mathbf{x}(n)$ again. If we apply this L times and use the periodicity conditions $\mathbf{v}(n+L) = \mathbf{v}(n)$ and $\mathbf{x}(n+L) = \mathbf{x}(n)$ we find

$$(\mathbf{I}-\mathbf{A}^L)\mathbf{v}(n) = \underbrace{[\mathbf{B} \quad \mathbf{A}\mathbf{B} \quad \ldots \quad \mathbf{A}^{L-1}\mathbf{B}]}_{\mathcal{R}_{A,B}(L)} \begin{bmatrix} \mathbf{x}(n-1) \\ \mathbf{x}(n-2) \\ \vdots \\ \mathbf{x}(n-L) \end{bmatrix}$$

Here we have used the notation that for any integer $i > 0$,

$$\mathcal{R}_{A,B}(i) \triangleq [\mathbf{B} \quad \mathbf{A}\mathbf{B} \quad \ldots \quad \mathbf{A}^{i-1}\mathbf{B}] \qquad (3)$$

Let N denote the state dimension (size of $\mathbf{v}(n)$) and r the number of inputs (size of $\mathbf{x}(n)$). Then $\mathcal{R}_{A,B}(i)$ is a $N \times ir$ matrix with rank $\leq N$. The matrix $\mathcal{R}_{A,B}(L)$, in particular, has size $N \times Lr$. Assume $(\mathbf{I}-\mathbf{A}^L)$ is nonsingular for reasons explained earlier. It is then clear that we can attain any value for the state $\mathbf{v}(n)$ at any time n by application of a suitable input $\mathbf{x}(n-1), \mathbf{x}(n-2), \ldots, \mathbf{x}(n-L)$ if and only if the matrix $\mathcal{R}_{A,B}(L)$ has rank N. This gives a test for reachability. Now two cases should be distinguished:

1. Let $N \leq L$. Then the rank of $\mathcal{R}_{A,B}(N) =$ rank of $\mathcal{R}_{A,B}(L)$ (Cayley-Hamilton theorem), and the reachability test reduces to the conventional one. Moreover, in the nonreachable case we can perform the usual state reduction.

2. Let $N > L$. This is possible in the mimo case (e.g., if $\mathbf{H}(k) = W_L^k\mathbf{I}_r$, then $N = r$ regardless of L). For this case two subcases are possible:

   (a) The rank of $\mathcal{R}_{A,B}(L)$ is already N, so the system is reachable.

   (b) The rank of $\mathcal{R}_{A,B}(L) <$ rank of $\mathcal{R}_{A,B}(N)$. If the latter is also $< N$, we can perform the usual reduction and reduce the size N of the state vector. If the rank of $\mathcal{R}_{A,B}(N)$ already N, we cannot do this, but we might still be able to perform a reduction of the cyclic state space equations as we shall demonstrate below.

2.2. Observability

State-observability in a cyclic LTI system can also be defined similar to the traditional case, but with some subtle distinctions between the cases $N \leq L$ and $N > L$. First assume $N \leq L$. The output equation $\mathbf{y}(n) = \mathbf{C}\mathbf{v}(n) + \mathbf{D}\mathbf{x}(n)$ can be repeatedly applied to yield

$$\begin{bmatrix} \mathbf{y}(n) \\ \mathbf{y}(n+1) \\ \vdots \\ \mathbf{y}(n+N-1) \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A} \\ \vdots \\ \mathbf{C}\mathbf{A}^{N-1} \end{bmatrix}}_{\mathcal{S}_{C,A}(N)} \mathbf{v}(n) + \mathbf{f} \qquad (4)$$

where $\mathbf{f}$ depends on $\mathbf{x}(n), \mathbf{x}(n+1), \ldots \mathbf{x}(n+N-1)$. The initial state $\mathbf{v}(n)$ can be uniquely found from the N samples of the input and output in this equation, as long as the matrix $\mathcal{S}_{C,A}(N)$, which has N columns, has rank N. If $N > L$, the preceding equation is not meaningful because $\mathbf{y}(i)$ and $\mathbf{x}(i)$ repeat with period L. In this case, however, we have a very unusual situation. If the input and output are known for all L values of time, then in particular $\mathbf{x}(i)$ is known for all i and we can identify the state $\mathbf{v}(n)$ for all n using the state recursion. Thus the notion of observability becomes trivial for $N > L$.

Example 1

Consider the cyclic system

$$H(k) = 1 + aW_L^k + a^2W_L^{2k} + \ldots + a^{L-1}W_L^{(L-1)k},$$

for which a direct-form implementation is shown in Fig. 4(a). With state variables as indicated, the state space description $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ can readily be identified, yielding

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{C} = [a^{L-1} \quad a^{L-2} \quad \ldots a] \qquad D = 1$$

Note that the number of state variables $N = L-1$. From the preceding we verify that

$$\mathcal{R}_{A,B}(L) = \begin{bmatrix} 0 & 0 & \ldots & 0 & 1 & 0 \\ 0 & 0 & \ldots & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 1 & \ldots & 0 & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{S}_{C,A}(N) = \begin{bmatrix} a^{L-1} & a^{L-2} & \ldots & a \\ 0 & a^{L-1} & \ldots & a^2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a^{L-1} \end{bmatrix}$$

Since $N = L-1$, $\mathcal{R}_{A,B}(L)$ has size $(L-1) \times L$ and $\mathcal{S}_{C,A}(N)$ has size $(L-1) \times (L-1)$. Both of these matrices have rank $N = L-1$ (assuming, of course, $a \neq 0$), showing that the structure is both reachable and observable. Notice, however, that the system $H(k)$ can be rewritten in the recursive form

$$H(k) = \frac{1-a^L}{1-aW_L^k}$$

using the fact that $W_L^L = 1$. This yields the simpler recursive implementation requiring only one cyclic delay W_L^k (Fig. 4(b)). We can verify that the state space description of the simplified structure is

$$A = a, \quad B = 1, \quad C = a(1-a^L), \quad D = 1 - a^L$$

In this case the number of state variables $N = 1$. One readily verifies that $\mathcal{R}_{A,B}(1) = 1$ and $\mathcal{S}_{C,A}(1) = a(1-a^L)$. So $\mathcal{R}_{A,B}(L)$ and $\mathcal{S}_{C,A}(N)$ have rank N, and the sturcture is reachable and observable (assuming $a \neq 0$ and $a^L \neq 1$). Thus the two structures shown in Fig. 4 are two reachable and observable implementations of $H(k)$ with different state dimensions! The first one requires $L-1$ cyclic delays (W_L^k elements); the second structure requires only one cyclic delay.

Example 2

Consider the 2×2 cyclic system shown in Fig. 5(a), and assume $L = 3$. The number of state variables is $N = 4$. The state space description has

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Let $L = 3$. Then explicit computation shows that

$$\mathcal{R}_{A,B}(L) = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{R}_{A,B}(N) = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{S}_{C,A}(2) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Thus $\mathcal{R}_{A,B}(L)$ has rank $3 < N$ which shows that the cyclic system is not reachable. However, $\mathcal{R}_{A,B}(N)$ has rank 4. Since $\mathcal{S}_{C,A}(2)$ has rank 4, so does $\mathcal{S}_{C,A}(N)$. So we cannot perform state-reduction using classical techniques. In this example, however, it is possible to perform state reduction of the cyclic system by simple manipulations of the structure, and by using the fact that $W_L^3 = 1$. For this we notice the identity

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & x \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = (1+x) \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

which shows that the transfer matrix of Fig. 5(a) is eventually

$$\begin{aligned} \mathbf{H}(k) &= \begin{bmatrix} 1 + W_3^{2k} & 0 \\ 0 & W_3^{2k} + W_3^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 + W_3^{2k} \\ W_3^k + W_3^{2k} \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} \end{aligned}$$

which has the implementation shown in Fig. 5(b) requiring only two cyclic delays. Thus in this example, $\mathcal{R}_{A,B}(N)$ and $\mathcal{S}_{C,A}(N)$ have rank N but $\mathcal{R}_{A,B}(L)$ does not, and we were able to reduce state-dimension.

In Example 1 we found that the state dimension could be reduced even though the cyclic system is reachable as well as observable. In Example 2 we found that $\mathcal{R}_{A,B}(N)$ and $\mathcal{S}_{C,A}(N)$ have rank N and $\mathcal{R}_{A,B}(L)$ has deficient rank, and the state dimension could again be reduced. The question now is, what is a necessary and sufficient condition for the minimality of state dimension in cyclic LTI structures? A related question is, can we develop a theory paralleling the Smith-McMillan form and relate the minimum state dimension (McMillan degree) to this form? These appear to be fundamental questions requiring further work.

2.3. Unitariness of Realization Matrix

Suppose we are given an implementation for a cyclic transfer matrix $\mathbf{E}(k)$. This implementation has a state space description of the form (1). The realization matrix for the implementation is defined as

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}$$

The following result proved in **[10]** connects the cyclic-paraunitary property to unitariness of the realization matrix.

Lemma 1. If the realization matrix is unitary, then the cyclic system $\mathbf{E}(k)$ is paraunitary. $\diamond$

3. CONCLUDING REMARKS

We conclude by making explicit the connection between three related problems in cyclic LTI system theory:

*1. **Paraunitary interpolation problem.*** Given a sequence of unitary matrices $\mathbf{E}(k), 0 \leq k \leq L-1$, does there exists an FIR paraunitary matrix

$$\mathbf{E}_{int}(z) = \sum_{n=0}^{N} \mathbf{e}_{int}(n) z^{-n}$$

such that $\mathbf{E}(k) = \mathbf{E}_{int}(W_L^{-k})$? This is called the paraunitary interpolation problem. In **[10]** it has been shown that such an interpolant $\mathbf{E}_{int}(z)$ does not always exist.

*2. **Cyclic paraunitary factorization problem.*** We know that any noncyclic causal FIR paraunitary system can be factorized into degree one building blocks $\mathbf{I} - \mathbf{u}_i\mathbf{u}_i^\dagger + \mathbf{u}_i\mathbf{u}_i^\dagger z^{-1}$ (where $\mathbf{u}_i$ are unit norm vectors). Can a cyclic paraunitary system $\mathbf{E}(k)$ be factorized into degree-one cyclic building blocks $\mathbf{U}_i(k) = \mathbf{I} - \mathbf{u}_i\mathbf{u}_i^\dagger + \mathbf{u}_i\mathbf{u}_i^\dagger W_L^k$? It turns out that this is not always possible **[10]**.

*3. **Unitary realization-matrix problem.*** Lemma 1 is analogous to a result in the noncyclic case **[8]**. However, unlike in the noncyclic case, we do not have the converse result. That is, even if $\mathbf{E}(k)$ is paraunitary, there may not exist a minimal nonrecursive structure (i.e., minimal structure with all eigenvalues of $\mathbf{A}$ equal to zero), with unitary realization matrix. When such a structure does exist, the FIR interpolant $\mathbf{E}_{int}(z) = \mathbf{D} + \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$, obtained by replacing W_L^k with z^{-1} in the structure, would be paraunitary (because the converse part holds in the noncyclic case [8]). Since cyclic paraunitary systems do not necessarily have FIR interpolants, this shows that $\mathbf{E}(k)$ does not always have a structure with unitary realization matrix.

By combining the preceding arguments we can show this: Let $\mathbf{E}(k)$ be cyclic paraunitary. Then the following three statements are equivalent: (a) there exists a causal FIR paraunitary interpolant $\mathbf{E}_{int}(z)$, (b) $\mathbf{E}(k)$ can be factorized into unitary building blocks like $\mathbf{U}_i(k)$ (and a constant factor representing $\mathbf{E}(0)$), and (c) there exists a cyclic recursive implementation for $\mathbf{E}(k)$ such that the realization matrix is unitary.

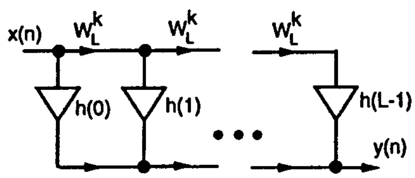

Fig. 1. Implementation of an arbitrary cyclic(L) LTI system.

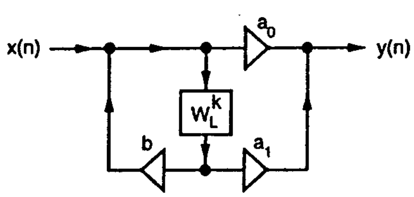

Fig. 2. The cyclic direct-form structure for a first order filter.

References

[1] Bopardikar, A. S., Raghuveer, M. R., and Adiga, B. S. "Perfect reconstruction circular convolution filter banks and their applications to the implementation of bandlimited discrete wavelet transforms", ICASSP, pp. 3665–3668, Munich, May 1997.

[2] Bopardikar, A. S., Raghuveer, M. R., and Adiga, B. S. "PRCC filter banks: theory, implementation, and application", Proc. SPIE., vol. 3169, San Diego, CA, July-Aug. 1997.

[3] Caire, G., Grossman, R. L., and Poor, H. V. "Wavelet transforms associated with finite cyclic groups," *IEEE Trans.IT.*, pp. 1157–1166, July 1993.

[4] Gold, B. and Rader, C. M. *Digital processing of signals*, McGraw Hill Book Co., NY, 1969.

[5] Chen, C. T. *Linear system theory and design*, Holt, Rinehart, and Winston, Inc., 1984.

[6] Kiya, H., Nishikawa, K., Sagawa, M. "Property of circular convolution for subband image coding," IEICE Trans. Fundamentals, pp. 852-860, July 1992.

[7] Smith, M. J. T., and Eddins, S. L. "Analysis/synthesis techniques for subband image coding," ICASSP, pp. 1446-1456, Aug. 1990.

[8] Vaidyanathan, P. P. *Multirate systems and filter banks*, Prentice Hall, 1993.

[9] Vaidyanathan, P. P., and Kirac, A. "Theory of cyclic filter banks" ICASSP, pp. 2449–2452, Munich, May 1997.

[10] — "Cyclic LTI systems and the paraunitary interpolation problem", ICASSP, Seattle, May 1998.

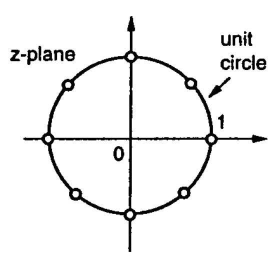

Fig. 3. The points on the unit-circle corresponding to the DFT frequencies (L=8).

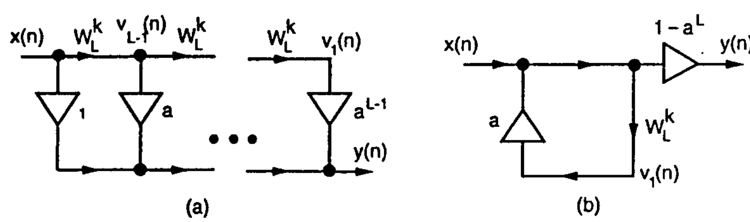

Fig. 4. Example 1. Two implementations of a cyclic(L) system. Both of these are reachabale and observable implementations. (a) L-1 state variables used and (b) only one state variable used.

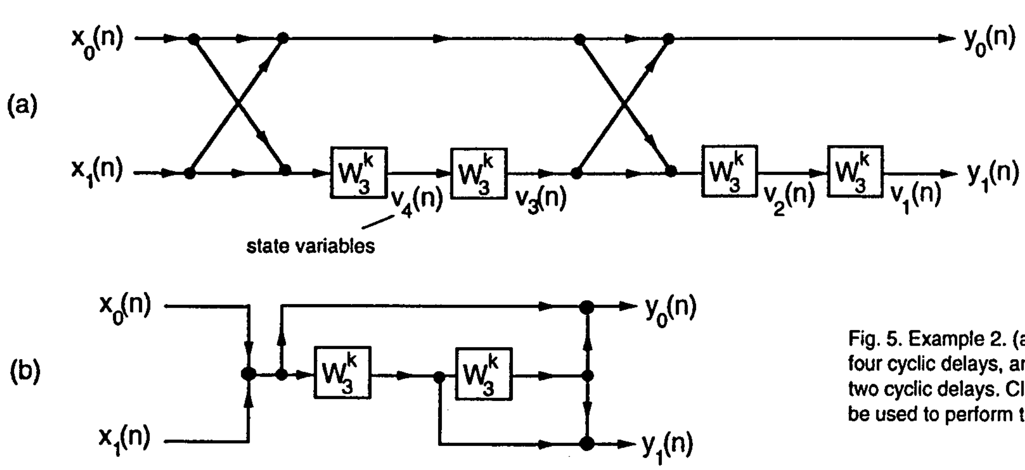

Fig. 5. Example 2. (a) A cyclic(3) system requiring four cyclic delays, and (b) reduced system requiring two cyclic delays. Classical techniques cannot be used to perform this reduction.